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On the growing up problem for semilinear heat equations

by

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§1. Introduction. This report is an extract of the joint paper by K. Kobayashi, T. Sirao and H. Tanaka [5], and the proofs of our theorems will be given in [5].

Let us consider the following Cauchy problem:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), \quad t > 0, \quad x \in R^d, \\ u(0,x) = a(x), \end{cases}$$

where Δ denotes Laplacian differential operator, f is a non-negative locally Lipschitz continuous function and a is a bounded non-negative continuous function. In this report, we limit the class of solutions of (1) as follows:

Definition 1.1. $u(t,x)$ is said to be a positive solution of (1) if there exists positive T_∞ ($\leqq \infty$) with the following properties (i), (ii) and (iii).

(i) For any positive $T < T_\infty$, $u(t,x)$ is bounded and continuous on $[0,T] \times R^d$.

(ii) $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j = 1,2,\ldots,d)$ exists in $(0,T_\infty) \times R^d$ and $u(t,x)$ satisfies (1) in the classical sense.

(iii) $u(t,x) > 0$ in $(0,T_\infty) \times R^d$.

Though small $T_\infty > 0$ always satisfies the above conditions, we will take $T_\infty = T_\infty(a,f)$ as the supremum of T_∞ satisfying (i)-

(iii). Then T_∞ may or may not be infinity. If $T_\infty = \infty$, then u is said to be a global solution. Otherwise u is a local solution.

The purpose of this report is to consider "How does the behavior of f near the origin effect to the growth of positive solution as $t \longrightarrow \infty$?" The answer will be given in §2.

When $f(u) = u^{1+\alpha}$, $\alpha > 0$, this problem was first considered by H. Fujita [1]. The main result in [1] is stated as follows: If $\alpha d < 2$, then all the positive solutions of (1) blow up in finite times, that is, there is no global solution of (1) for any non-trivial $a(x) \geqq 0$. On the contrary, if $\alpha d > 2$ then there exist global solutions for small $a(x) \geqq 0$. About the critical case where $\alpha d = 2$, H. Hayakawa [3] proved the non-existence of global solution for non-trivial $a(x) \geqq 0$. Then S. Sugitani [6] considered Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = -(-\Delta)^\beta u + u^{1+\alpha}, \quad t > 0, \quad x \in R^d,$$

where $0 < \beta < 1$, and obtained the same conclusion for $\alpha d \leqq 2\beta$.

On the other hand, Ya. I. Kaneli [4] discussed related problems about (1) in 1-dimensional case. Among others, he says that if

(2) $f(0) = f(1) = 0, \quad f(u) > 0, \quad 0 < u < 1, \quad f'(0) > 0,$

(3) $a(x) > 0$ on a certain interval and $0 \leqq a(x) \leqq 1$ everywhere,

then the solution $u(t,x)$ of (1)—(3) converges to 1 uniformly on every finite interval as $t \to \infty$. (Though another interesting results are stated in [4], they are slightly different from our present interest.)

§2. Results. Before stating our results, we give notations and terminologies.

$\mathcal{F}$ denotes the class of all functions satisfying the following conditions (A) and (B).

(A) f is a locally Lipschitz continuous function on $[0,\infty)$ and $f(0) = 0$, $f(u) > 0$ for $u > 0$.

(B) There exists a positive constant c_0 such that

$$f(uv) \geqq c_0 v^{1+\frac{2}{d}} f(u) \quad \text{for} \quad 0 \leqq u \leqq v, \quad u < c_0 \quad \text{and} \quad uv < c_0 .$$

$\tilde{\mathcal{F}}$ is the class of all non-decreasing functions f satisfying (A) and (C) stated below.

(C) There exists a positive constant c such that

(a) $f(uv) \geqq cv^{1+\frac{2}{d}} f(u)$ for $0 < u \leqq v$, $u < c$,

(b) $f(uv) \geqq cv^{2+\frac{2}{d}} f(u)$ for $0 < v \leqq u < c$.

Obviously $\tilde{\mathcal{F}} \subset \mathcal{F}$. (cf. Remark 3.)

Definition 2.1. $T_\infty = T_\infty(a,f)$ in §1 is said to be the blowing up time of the solution of $u(t,x)$ of (1). (i) If T_∞ is finite, then we say that $u(t,x)$ blows up (in finite time). (ii) If $T_\infty = \infty$ and $u(t,x) \to \infty$ uniformly in x on every compact set as $t \to \infty$, then we say that $u(t,x)$ grows up to infinity.

The solution of (1) corresponding to f and a is denoted by $u(t,x;\, a,f)$.

Now we can state the following

Theorem 1. Let $f \in \tilde{\mathcal{F}}$. If, for any $\varepsilon > 0$,

(4) $$\int_0^{\varepsilon} f(u)/u^{2+\frac{2}{d}}\, du = \infty,$$

then the positive solution $u(t,x;\, a,f)$ of (1) blows up for any non-trivial $a(x) \geqq 0$.

Theorem 2. Let $f \in \tilde{\mathcal{F}}$. (i) If (4) holds for any $\varepsilon > 0$, then any positive solution $u(t,x;\, a,f)$ of (1) blows up or grows up to infinity. (ii) If the left hand side of (4) is finite for a certain $\varepsilon > 0$, then, for small initial data $a(x) = \alpha e^{-\beta|x|^2} > 0$, the solution $u(t,x;\, a,f)$ of (1) converges to 0 uniformly in x as $t \to \infty$.

Theorem 3. Let f be a Lipschitz continuous function on $[0,1]$ such that $f(u) > 0$ for $0 < u < 1$ and $f(0) = f(1) = 0$. Moreover we assume that f satisfies the conditions (B) and (4). Then, for each continuous initial data $a(x)$ with $0 \leqq a(x) \leqq 1$, $a(x) \not\equiv 0$, the solution $u(t,x;\, a,f)$ of (1) converges to 1 uniformly on every compact set ($\subset R^d$) as $t \to \infty$.

Remark 1. The assumptions of (ii) in Theorem 2 can be weakend as follows: f is a locally Lipschitz continuous function satisfying (iia) $f(u) \geqq 0$ and $f(0) = 0$, (iib) $f(uv) \geqq vf(u)$ for $u \geqq 0$, $v \geqq 1$, and (iic) the left hand side of (4) is finite.

Remark 2. For each $f \in \mathcal{F}$ satisfying (4), there exists $\tilde{f} \in \tilde{\mathcal{F}}$ such that (4) holds for $\tilde{f}$ and

$$\liminf_{u \downarrow 0} \frac{f(u)}{\tilde{f}(u)} > 0.$$

Remark 3. As an application of Theorem 2, let us consider the case when f is given by

$$f(u) = u^{1+\frac{2}{d}}\{\log \frac{1}{u} \log_{(2)} \frac{1}{u} \cdots \log_{(n-1)} \frac{1}{u} (\log_{(n)} \frac{1}{u})^{\delta}\}^{-1}$$

near the origin and smooth and positive in the whole of $(0,\infty)$, where $\delta > 0$ and $\log_{(k)} u = \log \log \ldots \log u$ (k-times). If $0 < \delta \leqq 1$, then $f \in \mathcal{F}$ and hence any positive solution of (1) blows up or grows up to infinity by Theorem 2, (i). On the other hand, if $\delta > 1$, then some positive solution $u(t,x)$ of (1) converges to 0 uniformly in x as $t \to \infty$ by Theorem 2, (ii).

Remark 4. The conditions (B) and (4) of Theorem 1 are concerned with the local behavior of f near the origin only apart from $f(u) > 0$ $(u > 0)$, while the condition (a) of (C) is concerned with the behavior of f for large u, that is, it implies that

(5) $\quad f(u) > \text{const.}\, u^{1+\frac{2}{d}}$ for all sufficiently large u.

Some condition on the behavior of $f(u)$ for large u such as (5) is required for the blowing up conclusion. This aspect will be made much clear by the following theorem which is a slight extension of one of results due to Fujita [2].

Theorem 4. Assume that

(i) $\quad \displaystyle\int^{\infty} \frac{d\lambda}{f(\lambda)} < \infty,$

(ii) there exist constants $c > 0$ and $u_0 > 0$ such that

$$f(u) \geqq cf(v) \quad \text{for} \quad u_0 < v < u.$$

Let $u(t,x)$ be a positive solution of (1). If for any $M > 0$ there exists $t_M > 0$ such that $u(t_M,x) > M$ for $|x| < 1$, then

$u(t,x)$ blows up.

Remark 5. The following two theorems were used to prove Theorem 1-3.

Theorem 5. Let f, $\tilde{f}$ be locally Lipschitz continuous functions on $[0,\infty)$, and assume that (i) $f(u) > 0$ for $u > 0$, (ii) $\tilde{f}$ is non-decreasing with $\tilde{f}(0) = 0$, and (iii)

$$\liminf_{u \downarrow 0} \frac{f(u)}{\tilde{f}(u)} > 0.$$

Suppose that, for each bounded continuous initial data $a(x) \geqq 0$, the solution $u(t,x;\, a,\tilde{f})$ of (1) either blows up or satisfies

$$\limsup_{t \to \infty} \| u(t,x;\, a,\tilde{f})\|_{\infty} = \infty,$$

where $\| \ \|_{\infty}$ denotes the supremum norm. Then any positive solution $u(t,x;\, a,f)$ of (1) blows up or grows up to infinity.

Theorem 6. Let f be a Lipschitz continuous function on $[0,1]$ such that $f(u) > 0$ for $0 < u < 1$ and $f(1) = 0$, and let f satisfy the same assumptions as in Theorem 5. Moreover we assume that, for any non-negative bounded continuous $a(x) \geqq 0$ ($\not\equiv 0$), the solution $u(t,x;\, a,\tilde{f})$ of (1) has the same property as in Theorem 5. Then, for any continuous function $a(X)$ with $0 \leqq a(x) \leqq 1$, $a(x) \not\equiv 0$, the solution $u(t,x;\, a,f)$ of (1) converges to 1 uniformly on each compact set of R^d as $t \to \infty$.

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